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The Impact of Additional Insurance Options of the ABD, ALI and ADI – Type on a Level of Mathematical Reserves of Premiums in Life Insurance

Pursuant to the Act, The insurance company shall have at its disposal a proper amount of reserves connected with an insurance premium, which shall suffice to cover the insurer's liabilities on account of reimbursement of future benefits. Methods of calculating mathematical reserves in traditional insurance may be found in a classical actuarial literature according to which a reserve is calculated as an actuarial value of accumulated future money flows including the risk of death and the change of money value in time, that is the so-called actuarial risk. Insurance companies offering complex insurance products such as life insurance with additional options, among other things, with the ADBs (Accelerated Death Benefits), ALIs (Acceleration Life Insurance), AI (Accident Insurance) and ADIs (Accidental Death Insurance) option, pursuant to SOLVENCY II ought to take into consideration in their calculations also an additional aspect of risk arising from extended actuarial risk which is covered. In this article by combining a financial and insurance attitude, calculation of reserves for a life insurance with additional option of the ADBs, ALIs and ADIs-type is made, which is determined as a proper conditional expected value with allowance for an extended actuarial risk and the impact of additional options on their amount is examined.

Keywords: multi-option life insurance, additional insurance options of the ABDs, ALI and ADI-type, solvency, mathematical reserves of premium

Introduction

Insurance business due to its social and economic significance has been subject to a specific supervision and on account of this many requirements are imposed on insurance companies, the objective of which is to assure their solvency and ensure the safety of the insured. A key aspect of the regulatory framework within this scope involves the necessity to determine the so-called increased risk capital, which shall suffice to cover an actual insurance risk covered by protection. In order to do that, the insurer, who wishes to protect himself against fortuitous losses based on valuation of money flows, ought to determine a proper level of reserves, which shall balance the risk he incurs. Pursuant to the Act[1], the insurance company shall have at its disposal, among other things, a proper amount of reserves connected with insurance premium, which is called a mathematical reserve of premiums and constitutes a sum saved up to cover future liabilities.

In order to achieve this goal, in case of complex insurance products which include a multi-option insurance, one ought to take into consideration an extended risk arising from additional options in calculations of the amount of reserves required. Pursuant to Solvency II[2], the best of assessment of reserves is a discounted value of all the future money flows of risks related. Therefore, the valuation of reserves executed on the basis of the best assessment of future money flows including discounting ought to be based on a market value of all the risks covered by contract, i.e. including filtration determining complete information available at the t-moment concerning the activation process of the options (ABD, ALI and ADI), which denotes the necessity to include a conditional expected value in calculations. Taking into account this aspect, in this article the analysis of required mathematical reserves of premiums is made that the insurance company, which offers life insurance with additional options of the ABD, ALI and ADI-type, ought to establish and the impact of the ABD, ALI and ADI options on their amount is examined.

1. A conception of insurance with additional options and their probabilistic structure

Life insurance and pure endowment insurance with the possibility of redemption of diverse additional options, which is called the multi-option insurance, is a contract concluded between the insured and the insurer, according to which the insurer undertakes to pay benefits on account of:

- a basic contract (covering a fundamental risk),
- additional contracts (covering an extended insurance risk).

The basic life and pure endowment insurance contract concerns two main life events: life and death. While, additional contracts concern life events distinguished within the first of them such as, for example, damage to health, disability, unfortunate accident, dread disease, incapacity to carry on a profession and others. From the point of view of the insured, the occurrence of a fortuitous event covered by the insurance contract denotes a life event, whereas from the point of view of the company, it denotes both a life event, and a proper active option of the policy which is called its state. In other words, each life event of the insured person matches a definite state from a finite set of possible states denoted as S, and a change in the life situation of the insured results in the change of status (state) of the insurance policy[3].

1. Ustawa z dnia 22 maja 2003r. o działalności ubezpieczeniowej (Dz. U. z 2003r. nr 124 poz. 1151).
2. Directive 2009/138/EC of the European Parliament and of the Council.
3. Bowers N.L., Gerber H.U.,Hickman J.C, Jones D.A., Nesbitt C., *Actuarial mathematics, The Society of Actuaries*, Schaumburg 1997, c.231.

At a specific moment of duration of the insurance contract only one insurance option is active and it depends on a life event of the insured person. The change in the life situation of the insured (the change of the life event) results in implementation of an adequate policy option and the change of its status, so a model describing dynamics of changes in life situations of the insured person is simultaneously the model describing a dynamic nature of activation of possible policy options. The most simple probabilistic model of insurance with an additional option includes three states corresponding to the following life events[4]:

- H – the insured is healthy,
- S – the insured is sick,
- D – the insured died.

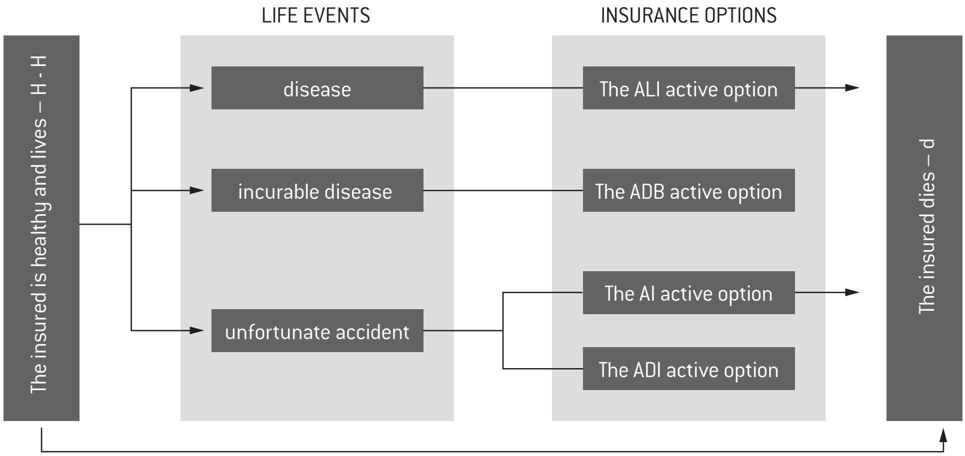
Such three-state probabilistic model in the presented thesis is generalised and applied in description of multi-option policies embracing the following additional options:

1. **ALI** – acceleration life insurance is an accelerated life insurance, where a part of a death benefit most frequently around 25% is paid to the insured to cover medical costs,
2. **ADB** – accelerated death benefits is the option of an accelerated benefit on account of death to cover medical costs for persons with a diagnosed incurable disease in terminal state of disease (the condition of taking advantage of the option is an expected future lifetime no longer than 12 months). Implementation of such option terminates the insurance,
3. **AI** – accident insurance is the option of accident insurance as a result of unfortunate accident,
4. **ADI** – accidental death insurance is death insurance as a result of unfortunate accident. Implementation of such option terminates the insurance.

The two first insurance options are the options from the sickness group, while the last two ones are the example of accident options. A pattern of possible life events covered by insurance and related options and transitions among them, that is, the probabilistic model of policy with the additional ALI, ADB, AI and ADI options are presented in Figure 1.

4. Wolthuis H., Life insurance mathematics (The Markovian model), Amsterdam: Universiteit van Amsterdam 2003, c.8.

Figure 1 A pattern of states and transitions among them in insurance with additional options of the ALI, ADB, AI and ADI-type



Source: own elaboration.

The $X(t)$ function is used to describe the change of states from the moment of contracting insurance, where $t \in T$ denotes time that passed from commencement of the insurance contract. Then, $X(t)$ denotes the state of the process, in other words, the policy status and an active insurance option at the t-moment of duration of the contract and the $X(t) = i$ notation denotes that at the t-moment, the X process is in the i-position, that is, the i-nth option of the contract is active. The life events happening in the life of the insured correspond to a concrete implementation of the process of activation of the $\{X(t)\}_{t \in T}$ · option. Let's observe that each life event is a random event, so for each t-moment, $X(t)$ is a random variable and $\{X(t)\}_{t \in T}$ is a stochastic process assuming values from the S finite space of states. The stochastic processes which enable to describe the process accompanying a multi-option insurance are the Markov processes. The Markov process in this case is a family of the $\{X(t); t \in T\}$ random variables determined on the common probabilistic space (Ω, F, P)[5].

The probability of conditional transition provided all possible life events, which happened in the past. A set of all possible fortuitous events, which happened at the t-moment, is denoted as F_t and it is called the history of activation process of options. In case of the Markov processes, a conditional probability does not depend on the entire history of the $\{X(t)\}$ process but merely it depends on its current state, then probability of transition is determined by means of the following formula[6]:

$$ {}_t p_x^{jk} = P(X(x+t) = k \mid X(x) = j) $$

where x – the age of the insured at the moment of conclusion of the contract.

Taking into account the actuarial notation, the following notation of probability of a change of status of a multi-option policy with the ALI, ADB, AI and ADI options, is assumed in the article:

${}_x p_t^{HH}$ – the probability that a healthy person at the age of x will also be healthy at the age of $x+t$, that is, at the t-moment of duration of the contract,

5. Habermann S., Pitaco E., *Actuarial models for disability insurance*, Chapman & Hall CRC, London 1999, c.2.
6. Ibidem, c.13.

${}_x p_t^{ALI}$, ${}_x p_t^{ADB}$ – the probability that a healthy person at the age of x will be sick and take advantage of the ALI or ADB option respectively at the age of $x+t$,

${}_x p_t^{AI}$, ${}_x p_t^{ADI}$ – the probability that a healthy person at the age of x will meet with an accident and take advantage of the AI or ADI option at the age of $x+t$,

${}_x p_t^{AI/AI}$ – the probability that a sick person taking advantages of the AI option at the age of x still takes advantages of such option at the age of $x+t$,

${}_x p_t^{ALId}$, ${}_x p_t^{AId}$ – the probability of death at the age of $x+t$ of a person who took advantage of the ALI or AI option,

${}_x p_t^{Hd}$ – the probability of death at the age of $x+t$ of a healthy person at the age of x,

Thus the probability of transition determined in this way for each one fulfills the Kolmogorov differential equations[7].

2. Payments stream and their actuarial value

Particular payment flows made by the insured (premium) and by the insurer (benefits) are connected with each insurance contract. Such payments form money flows between the insurer and the insured and determination of their amount is essential to carry out the assessment of multi-option policies. The $B(t)$ payment function in a multi-option insurance allows for all the payments corresponding to different life events happening in the life of the insured. In calculations concerning life insurance and additional insurance, the up-to-date value of money flows plays the essential role, that is, the so-called current equivalent of future payments. Valuation of the $B(t)$ financial stream, that is, the function of the up-to-date at the t-moment, value of payment flows made in the T-period of time is determined by the following formula[8]:

$$ZB_t\ \mathrm{T} = \frac{1}{v(t)_{\mathrm{T}}} \int v(\tau)\, dB(\tau)$$

The payment flow is determined by the $B(t)$ symbol, in case of multi-option insurance it ought to allow for all the payments corresponding to different life events happening in the life of the insured. Such payments may be divided into two groups. The first of them is connected with duration of activation process of options in a determined state. This type of payment covers:

- insurance premiums paid by the insured and annuities reimbursed by the insurer,
- benefits reimbursed by the insurer on account of survival till the end of the insurance period.

The second group of payments is connected with the change of state by the activation process of options. All the remaining benefits reimbursed on account of the death of the insured, death as a result of an unfortunate accident or permanent disablement belong to this group. They are reimbursed at the moment of the occurrence of a fortuitous event, so at any moment of the duration of insurance. The following markings of payments are used in this thesis:

- the amount of a premium paid by the insured at the t-moment, when the activation process of options is in the j state – $\pi_j(t)$

7. Iosifescu M., *Skończone procesy Markowa i ich zastosowania*, PWN, Warszawa 1988, c.57.
8. Wüthrich M.V., Bühlmann H., Furrer H., *Market-Consistent Actuarial Valuation*, Springer, New York 2007, c.21.

- the amount of annuity/single benefit paid by the insurer at the *t*-moment, when the activation process of options is in the state *j* is $r_j(t)/\ d_j(t)$,
- a single benefit connected with transition of the activation process of options from the *j* state to *k* at the *t*-moment is marked as $c_{jk}(t)$.

Then the up-to-date value of cumulated payments is a random process of the following form[9]:

$$ZB_t\,(T) = \frac{1}{v(t)}\sum_j \int_T v(\tau) r_j(\tau) I_{\{X(\tau)=j\}}\, d\tau + \frac{v(n)}{v(t)}\sum_j d_j(n) I_{\{X(n)=j\}} + \\ + \frac{1}{v(t)}\sum_j \sum_{k\neq j} \int_T v(\tau) c_{jk}(\tau) dN_{jk}(\tau) + \\ - \frac{1}{v(t)}\sum_j \int_T v(\tau) \pi_j(\tau) I_{\{X(\tau)=j\}}\, d\tau \qquad ,$$

where $N_{jk}(t) = \#\{\tau \in [0,t] : X(\tau - 0) = j, X(\tau) = k\}$.

3. Mathematical reserves of premiums versus the process of option activation

Pursuant to SOLVENCY II, the best assessment of reserves is probability weighted average of future money flows with allowance for the change of money value in time. In case of traditional insurance, when calculating mathematical reserves, filtration generated by the insurance portfolio is included and it is based on the process of mortality[10]:

$$F_t = \sigma\{I(T_i \le t),\ 0 \le t \le T,\ i=1, \ldots, l_x\}$$

Owing to that, mathematical reserves of classic life insurance and pure endowment insurance ought to be determined as the following conditional expected value[11]:

$$V_t(B,T) = E(ZB_t(T)\mid F_t) = E\left(\frac{1}{v(t)}\int_T v(\tau) dB(\tau)\mid F_t\right)$$

In case of multi-option insurance, status of the insurance policy determines type and amount of future benefits, so the insurer ought to allow for the history of activation process of options. Taking into consideration the up-to-date value of payments characteristic for multi-option insurance and applying the Markov processes to describe it, when calculating reserves the option of policy

9. Homa M., *Cena a ryzyko w wieloopcyjnych ubezpieczeniach na życie*, „Finanse w niestabilnym otoczeniu – dylematy i wyzwania, Studia Ekonomiczne", 2012 nr 109, 79–94.

10. Graf S., Kling A., Ruß J., *Risk analysis and valuation of life insurance contracts: Combining actuarial and financial approaches*, „Insurance: Mathematics and Economics", 2011 Vol.49, 115–125.

11. Błaszczyszyn B., Rolski T., *Podstawy matematyki ubezpieczeń na życie*, Wydawnictwo Naukowo-Techniczne, Warszawa 2004, c.185.

active at the *t*-moment ought to be included. Then a required level of mathematical reserve of premiums is determined in accordance with a forward-looking principle from the following formula[12]:

$$V_i(B,T) = E\Big(\frac{1}{v(t)}\sum_j \int_T v(\tau) r_j(\tau) I_{\{X(\tau)=j\}}\,\mathrm{d}\tau \,\Big|\, X(t)=i\Big) +$$
$$+ E\Big(\frac{v(n)}{v(t)}\sum_j d_j(n) I_{\{X(\tau)=j\}} \,\Big|\, X(t)=i\Big) +$$
$$+ E\Big(\frac{1}{v(t)}\sum_j \sum_{k\neq j} \int_T v(\tau) c_{jk}(\tau)\, dN_{jk}(\tau) \,\Big|\, X(t)=i\Big) +$$
$$- E\Big(\frac{1}{v(t)}\sum_j \int_T v(\tau) \pi_j(\tau) I_{\{X(\tau)=j\}}\,\mathrm{d}\tau \,\Big|\, X(t)=i\Big)$$

When calculating an expected value of the up-to-date payments and applying the actuarial notation, we obtain:

$$V_i(t,n) = \frac{1}{v(t)}\sum_j \int_t^{n\vee\infty} v(\tau) r_j(\tau)\, {}_{\tau-t}p^{ij}_{x+t}\,\mathrm{d}\tau +$$
$$+ \frac{v(n)}{v(t)}\sum_j d_j(n)\, {}_{n-t}p^{ij}_{x+t} +$$
$$+ \frac{1}{v(t)}\sum_j \sum_{k\neq j} \int_t^{n\vee\infty} v(\tau) c_{jk}(\tau) \mu_{jk}(x+\tau)\, {}_{\tau-t}p^{ij}_{x+t}\,\mathrm{d}x +$$
$$- \frac{1}{v(t)}\sum_j \int_t^{n\vee\infty} v(\tau) \pi_j(\tau)\, {}_{\tau-t}p^{ij}_{x+t}\,\mathrm{d}\tau$$

This is a final formula for a mathematical reserve of premiums of multi-option life insurance and pure endowment insurance.

4. Examples and results of applications

As the example, an n-year futures contract of basic life insurance, pure endowment insurance or a mixed life and endowment insurance is analysed (depending on the variant) along with additional contracts allowing for the ALI, ADB options concerning an accelerated sickness benefit and the ADI and AI options concerning results of unfortunate accidents. Money flows characteristic for life insurance and pure endowment insurance with the ALI, ADB, AI and ADI options, whose probabilistic structure illustrates a diagram in Fig. 1, is presented in Table 1.

12. Homa M., *Stochastyczne modele rezerw*, w: *Modele aktuarialne*, red. W.Ostasiewicz, Wydawnictwo UE, Wrocław 2000, c.84.

Table 1. Payment flows for the insurance with additional options: ALI, ADB, AI and ADI

Policy status	Premiums	Accident annuity	Benefit on account of survival to termination of insurance	Accelerated single benefit	Single benefit on account of accident	Single benefit on account of death
H	$\pi_H(t)$	–	$d_H(t)$	–	–	$c_{Hd}(t)$
ALI	$\pi_{ALI}(t)$	–	$d_{ALI}(t)$	$c_{ALI}(t)$	–	$c_{ALId}(t)$
ADB	–	–	–	$c_{ADB}(t)$	–	–
AI	–	$r_{AI}(t)$	$d_{AI}(t)$	–	–	$c_{AId}(t)$
ADI	–	–	–	–	$c_{ADI}(t)$	–

Source: own elaboration

The insurance subject of the basic contract is life of the insured, who will receive a reimbursement of the amount insured depending on a variant accepted on account of death during the insurance period or survival till the end of the insurance period. The insurance subject of the additional contract is health of the insured, and a lasting disability or a lasting partial disability, which is caused by an unfortunate accident, is covered by the scope of the AI and ADI options. In particular, this involves a reimbursement of annuity in respect of a lasting disability as a result of an unfortunate accident and the amount set out in the General Conditions of Insurance on account of death as a result of an unfortunate accident (most frequently this is a doubleness of the amount insured). While, the ALI and ADB options allow the insured to take advantage of the so-called accelerated benefit amounting to 25% and 100% of the amount insured respectively. The following payment flows are connected with insurance formulated in such manner:

1. Premiums paid during the insurance period:

$$\pi_H(t) = \pi_{ALI}(t) = \begin{cases} \pi & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases},$$

2. Annuity reimbursed on account of a lasting disability option:

$$r_{AI}(t) = \begin{cases} r & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases}$$

3. Benefit on account of an accelerated benefit option:

$$c_{ALI}(t) = \begin{cases} \alpha \times c & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases}, \quad c_{ADB}(t) = \begin{cases} c & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases}$$

4. Benefit on account of death of the insured as a result of an unfortunate accident:

$$c_{ADI}(t) = \begin{cases} \beta \times c & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases}$$

5. Benefit on account of death of the insured within the life insurance or life and endowment insurance:

$$c_{Hd}(t) = c_{AId}(t) = \begin{cases} c & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases}, \quad c_{ALId}(t) = \begin{cases} (1-\alpha)c & \text{for } 0 \le t < n \\ 0 & \text{for } t > n \end{cases}$$

6. Benefit on account of survival within the pure endowment insurance or life and endowment insurance:

$$d_H(t) = d_{AI}(t) = \begin{cases} d & \text{for } t = n \\ 0 & \text{for } t \ne n, \end{cases} \quad d_{ALI}(t) = \begin{cases} d - \alpha c & \text{for } t = n \\ 0 & \text{for } t \ne n \end{cases}$$

Actuarial values of the payment flows arising from the basic and additional contract for the insurance formulated in such manner are presented in Table 2.

Table 2. Actuarial value of future money flows for the insurance with additional values: ALI, ADB, AI and ADI

Policy status	Type of payment	Actuarial value
Premiums	$\pi_H(t) \approx \pi$ $\pi_{ALI}(t) \approx \pi$	$E(Z\pi_t \mid H) = \frac{\pi}{v(t)}\int_t^n v(\tau)\left({}_{\tau-t}p_{x+t}^{HH} + {}_{\tau-t}p_{x+t}^{ALI/ALI}\right)d\tau$ $E(Z\pi_t \mid ALI) = \frac{\pi}{v(t)}\int_t^n v(\tau)\,{}_{\tau-t}p_{x+t}^{ALI/ALI}\,d\tau$
Accident annuity	$r_{AI}(t) \approx r$	$E(ZR_t \mid H) = \frac{r}{v(t)}\int_t^n v(\tau)\,{}_{\tau-t}p_{x+t}^{AI}\,d\tau$ $E(ZR_t \mid AI) = \frac{r}{v(t)}\int_t^n v(\tau)\,{}_{\tau-t}p_{x+t}^{AI/AI}\,d\tau$
Benefit on account of survival to termination of insurance	$d_H(t) \approx d$ $d_{AI}(t) \approx d$ $d_{ALI}(t) \approx d - 25\%c$	$E(ZD_t \mid H) = d\,\frac{v(n)}{v(t)}\left({}_{\tau-t}p_{x+t}^{HH} + {}_{\tau-t}p_{x+t}^{AI} + {}_{\tau-t}p_{x+t}^{ALI}\right)$ $E(ZD_t \mid AI) = d\,\frac{v(n)}{v(t)}\,{}_{\tau-t}p_{x+t}^{AI/AI}$ $E(ZD_t \mid ALI) = (d - 0{,}25c)\,\frac{v(n)}{v(t)}\,{}_{\tau-t}p_{x+t}^{AI/AI}$
Single benefit on account of death	$c_{Hd}(t) \approx c$ $c_{AId}(t) \approx c$ $c_{ALId}(t) \approx 75\%c$	$E(ZC_t \mid H) = \frac{c}{v(t)}\int_t^n v(\tau)\mu(x+\tau)\left({}_{\tau-t}p_{x+t}^{HH} + {}_{\tau-t}p_{x+t}^{AI} + {}_{\tau-t}p_{x+t}^{ALI}\right)d\tau$ $E(ZC_t \mid AI) = \frac{c}{v(t)}\int_t^n v(\tau)\mu(x+\tau)\,{}_{\tau-t}p_{x+t}^{AI/AI}\,d\tau$ $E(ZC_t \mid AI) = \frac{0{,}75c}{v(t)}\int_t^n v(\tau)\mu(x+\tau)\,{}_{\tau-t}p_{x+t}^{ALI/ALI}\,d\tau$
Accelerated single benefit	$c_{ALI}(t) \approx 25\%c$ $c_{ADD}(t) \approx c$	$E(ZC_t \mid H) = \frac{c}{v(t)}\int_t^n v(\tau)\left(\sigma_{ADB}(x+\tau) + 0{,}25\sigma_{ALI}(x+\tau)\right){}_{\tau-t}p_{x+t}^{HH}\,d\tau$
Single benefit on account of accident	$c_{ADI}(t) \approx 2c$	$E(ZC_t \mid H) = \frac{2c}{v(t)}\int_t^n v(\tau)\sigma_{ADI}(x+\tau)\,{}_{\tau-t}p_{x+t}^{HH}\,d\tau$

Source: own elaboration

The actuarial value of future money flows in insurance with additional options depends on type of future payments and also the policy status, that is, an active option. Therefore, the insurer in order to cover future benefits shall have at its disposal a mathematical reserve of premiums equal to:

$$V(t,n) = \begin{cases} 0 & \text{when} \quad X(t) = ADB \vee X(t) = ADI \vee X(t) = d \\ V_H(t,n) & \text{when} \quad X(t) = H \\ V_{ALI}(t,n) & \text{when} \quad X(t) = ALI \\ V_{AI}(t,n) & \text{when} \quad X(t) = AI \end{cases}.$$

The additional ADB and ADI options and the d policy status are the so-called absorbing states, that is, they involve a termination of insurance protection and therefore the mathematical reserves of premiums are equal to zero. Whereas, if at the t-moment the insured is healthy and pays premiums, which means that the policy has the status marked as H, so a required level of mathematical reserves of premiums ought to be determined from the formula:

$$V_H(t,n) = \underbrace{\frac{c}{v(t)}\int_t^n v(\tau)\mu(x+\tau)\left({}_{\tau-t}p^{HH}_{x+t}+0{,}75\,{}_{\tau-t}p^{AI}_{x+t}+{}_{\tau-t}p^{ALI}_{x+t}\right)d\tau}_{E(\text{future benefits from a basic pure life insurance contract UZ})} +$$

$$\underbrace{+d\,\frac{v(n)}{v(t)}\left({}_{\tau-t}p^{HH}_{x+t}+{}_{\tau-t}p^{AI}_{x+t}+{}_{\tau-t}p^{ALI}_{x+t}\right)}_{E(\text{future benefits from a basic endowment insurance contract UD})}+$$

$$\underbrace{+\frac{r}{v(t)}\int_t^n v(\tau)\,{}_{\tau-t}p^{AI}_{x+t}d\tau+\frac{2c}{v(t)}\int_t^n v(\tau)\sigma_{ADI}(x+\tau)\,{}_{\tau-t}p^{HH}_{x+t}d\tau}_{E(\text{future benefits on account of consequences of accidents the AI and ADI options})}+$$

$$\underbrace{+\frac{c}{v(t)}\int_t^n v(\tau)\left(\sigma_{ADB}(x+\tau)+0{,}25\sigma_{ALI}(x+\tau)\right)\,{}_{\tau-t}p^{HH}_{x+t}d\tau}_{E(\text{future benefits on account of cacceleraaated options: ALI and ADB})}+$$

$$\underbrace{+\frac{\pi}{v(t)}\int_t^n v(\tau)\left({}_{\tau-t}p^{HH}_{x+t}+{}_{\tau-t}p^{ALI/ALI}_{x+t}\right)d\tau}_{E(\text{future premiums})}$$

Activation of the additional ALI and AI options impacts on the change of a level of mathematical reserves of premiums that the insurer shall have at its disposal at the *t*-moment. Depending on the options, they are equal appropriately:

$$V_{ALI}(t,n) = \underbrace{\frac{0{,}75c}{v(t)}\int_t^n v(\tau)\mu(x+\tau)\,{}_{\tau-t}p^{ALI/ALI}_{x+t}d\tau}_{E(\text{future life insurance benefits UZ})}+$$

$$\underbrace{+(d-0{,}25c)\,\frac{v(n)}{v(t)}\,{}_{\tau-t}p^{ALI/ALI}_{x+t}}_{E(\text{future survival benefits UD})}-\underbrace{\frac{\pi}{v(t)}\int_t^n v(\tau)\,{}_{\tau-t}p^{ALI/ALI}_{x+t}\,d\tau}_{E(\text{future premiums})}$$

and

$$V_{AI}(t,n) = \underbrace{\frac{r}{v(t)}\int_t^n v(\tau)\,{}_{\tau-t}p^{AI/AI}_{x+t}d\tau}_{E(\text{future annuity})}+\underbrace{\frac{c}{v(t)}\int_t^n v(\tau)\mu(x+\tau)\,{}_{\tau-t}p^{AI/AI}_{x+t}d\tau}_{E(\text{future life insurance benefits } UZ)}+\underbrace{d\,\frac{v(n)}{v(t)}\,{}_{\tau-t}p^{AI/AI}_{x+t}}_{E(\text{future pure endowment insurance benefits } UD)}$$

The formulae above constitute the basis of calculations of mathematical reserves of premiums arising from the insurance with the amount insured equal to 1000 monetary units and a risk-free rate equal to 5%. Moreover, to determine the probability of survival and death the Life Expectancy Tables is used and the μ $(x+\tau)$ function and the σ $(x+\tau)$ intensity of the occurrence of an unfortunate accident are based on the Gompertz-Makeham law[13], while as an incidence function of the most frequent malignant neoplasm in Poland, the Weibull hazard function marked as v $(x+\tau)$ is used. Estimators of the highest likelihood of parameters applied in the hazard function are presented in Table 3.

13. Dahl M., *Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts*, „Insurance: Mathematics and Economics", 2004 Vol.35, 113–136.

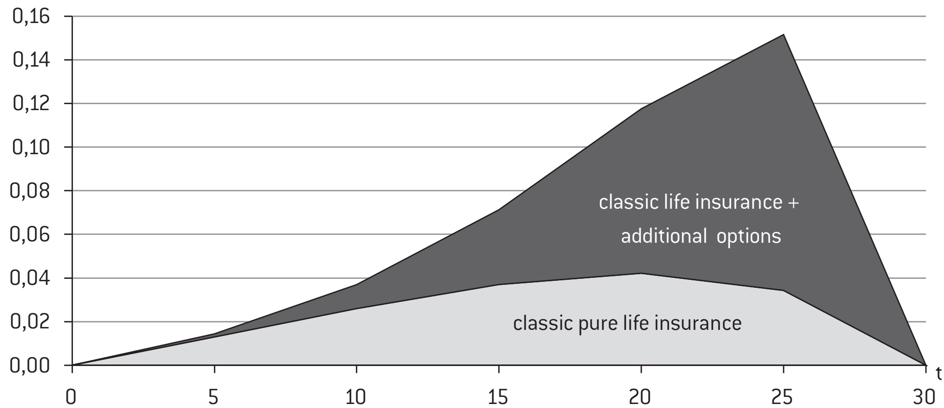
Table 3. Estimators of the highest likelihood and the Gompertz-Makeham and Weibull function

Parameters of a hazard function	$\mu(x+\tau)$	$\sigma(x+\tau)$	$\sigma_1(x+\tau)$	$\sigma_2(x+\tau)$
A	0,0000100	0,0004	0,026138636	0,080038684
B	0,000143493	3,4674E-06	1,300955033	1,089720724
C	1,0869039	1,148153621		

Source: own elaboration http://repozytorium.uni.lodz.pl (25.04.2015) oraz www.biecek.pl. (25.04.2015)

On that basis, when solving the Chapman-Kolmogorov differential equations, the probability of remaining in a state and transitions at any moment of the insurance period is determined is used in further calculations. The level of determined mathematical reserves for the type of pure endowment insurance and life insurance in a classic version and with allowance for a risk surplus of additional options are presented in pictures below[14].

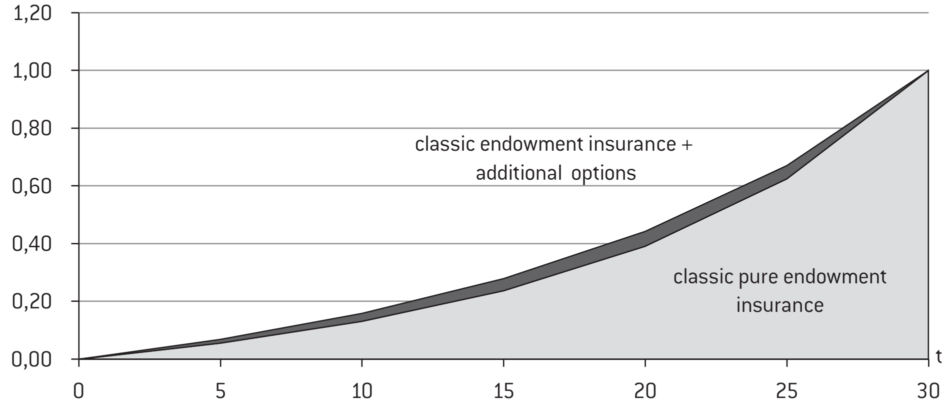
Figure 2 Reserve of premiums in a classic life insurance and with the ALI, ADB, AI, ADI options



Source: own elaboration.

14. Calculations were made with the use of Mathematica 8.0 Software.

Figure 3 Reserve of premiums in a classic pure endowment insurance and with the ALI, ADB, AI, ADI options



Source: own elaboration.

Fig.2–3 present graphs of functions of basic mathematical reserves of premiums that the insurer ought to have if it proposes traditional life insurance or pure endowment insurance and when it offers such insurance along with additional options.On the basis of the graphs above, one may find that the risk of additional options of the ALI, ADB, AI and ADI-type options does not impact on a functional form of mathematical reserves and does not change its structure during the insurance period, i.e. in case of a life insurance, the function of reserves is an increasing-decreasing time function of a zero final reserve, while in case of pure endowment insurance, this is an increasing function of a non-zero final reserve. Undoubtedly, the risk of options in a significant manner increases the amount of required, pursuant to Solvency II, mathematical reserves of premiums in case of life insurance, in case of pure endowment insurance the difference of a reserve level is not that big. The results of calculations explicitly confirm that the insurer offering the product in the form of a complex life insurance and endowment insurance has to accumulate a higher amount of basic reserves to secure its solvency in the future.

The changes of a level of reserves as a result of activation of additional options during the insurance period are also examined. The tables below present the amount of basic reserves $V_H(t,n)$ and the reserves that the insurer shall have at its disposal in case of activation of additional options, that is, $V_{ALI}(t,n)$ and $V_{AI}(t,n)$.

Table 4. The amount of a basic mathematical reserve of premiums and the ones connected with activation of options in multi-option life insurance (UZ)

czas	$V_H(t,n)$	$V_{ALI}(t,n)$	$V_{AI}(t,n)$
0	0,00000	0,0927052	0,123607
5	0,01434	0,0949038	0,126538
10	0,03691	0,095109	0,126812
15	0,07122	0,0911942	0,121592
20	0,11749	0,0794126	0,105884
25	0,15152	0,0530966	0,0707955
30	0,00000	0,00000	0,00000

Source: own elaboration

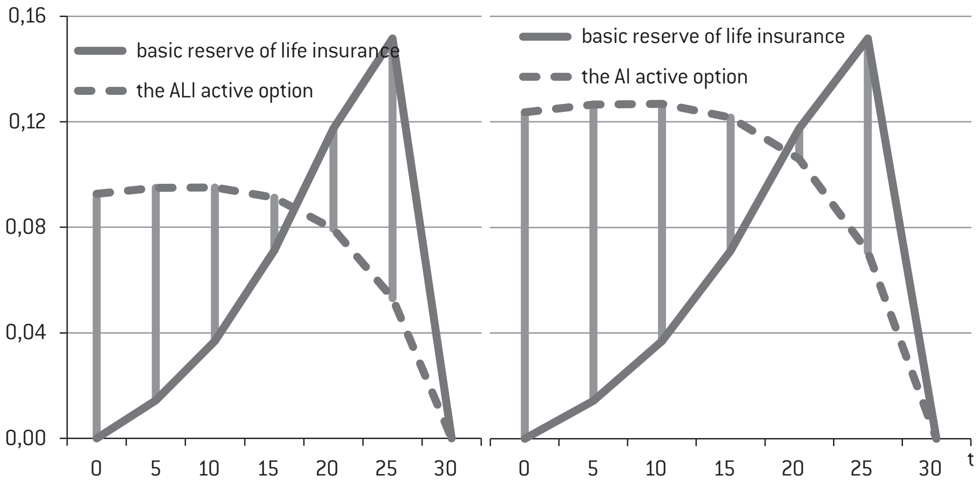
Table 5. The amount of a basic mathematical reserve of premiums and the ones connected with activation of options in multi-option pure endowment insurance (UD)

czas	$V_H(t,n)$	$V_{ALI}(t,n)$	$V_{AI}(t,n)$
0	0,0000	0,1281	0,1709
5	0,0686	0,1688	0,2251
10	0,1586	0,2231	0,2975
15	0,2791	0,2966	0,3955
20	0,4427	0,3978	0,5303
25	0,6706	0,5405	0,7207
30	1,0000	1,0000	1,0000

Source: own elaboration

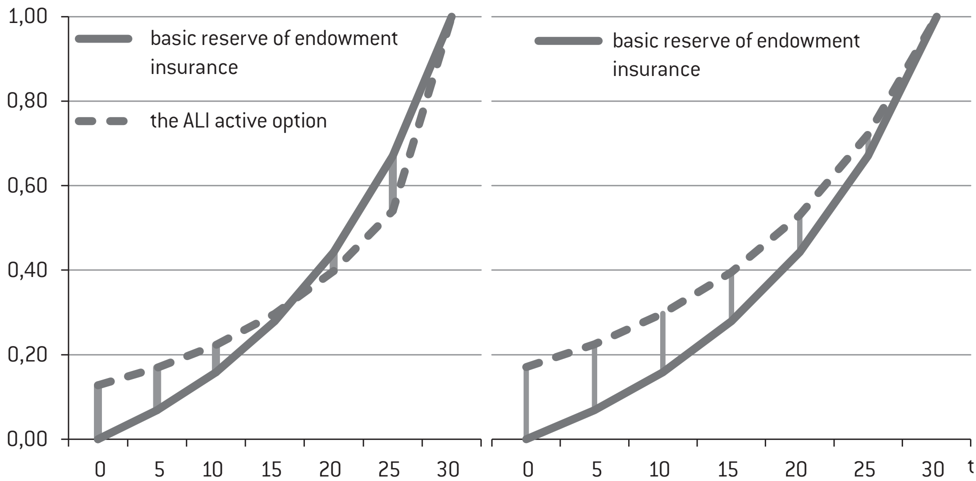
On the basis of the results above, one may find that when calculating mathematical reserves in case of multi-option contracts of a life-insurance-type and pure endowment ones, one ought to take into consideration the process of activation of options, which impacts on a level of required mathematical reserves of premiums. Depending on the type of a basic contract (a life or pure endowment insurance), one may observe a surplus or shortfall of reserves, which comes into being during the insurance period as a result of the change of policy status (activation of additional options), which is shown in Fig. 4.–5.

Figure 4 The amount of a shortfall/surplus of reserves arising from activation of additional options (ALI & AI).



Source: own elaboration.

Figure 5 The amount of a shortfall/surplus of reserves arising from activation of additional options (ALI & AI).



Source: own elaboration.

One ought to observe that activation of an additional option in life insurance in the first half of the insurance period requires involving additional sources and increasing reserves from the insurer, whereas in the second half of the insurance period we deal with a surplus of sources, which means that the insurer may release a part of mathematical reserve of premiums. We deal with an analogous situation in case of pure endowment insurance and activation of the option of the ALI-type. However, activation of an additional option of the AI-type increases the amount of mathematical reserves of premiums and requires involving additional sources to cover them during the entire insurance period.

5. Conclusion

On the basis of results obtained, one may find that irrespective of a type of a basic contract (life insurance, pure endowment insurance or a life and endowment insurance), an additional actuarial risk arising from additional options of the ALI, ADB, AI, ADI-type in a significant manner determines the amount of a mathematical reserve of premiums which is required pursuant to Solvency II. Owing to this, the process of activation options may not be omitted in valuations and calculations made aiming at assuring the solvency of the insurer. A proposed extension of basic formulae according to which one ought to determine both a basic mathematical reserve of premiums, and also its change triggered off by the change of a policy status, allows to determine a proper amount of reserves that the insurer offering complex insurance products shall have at its disposal.

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Wpływ dodatkowych opcji ubezpieczeniowych typu ADB, ALI i ADI na poziom rezerw matematycznych składek w ubezpieczeniach na życie

Zgodnie z ustawą Towarzystwo ubezpieczeniowe powinno dysponować odpowiednią wysokością rezerw związanych ze składką ubezpieczeniową, które wystarczą na pokrycie zobowiązań ubezpieczyciela z tytułu wypłaty przyszłych świadczeń. Metody obliczania rezerw matematycznych w tradycyjnych ubezpieczeniach można znaleźć w klasycznej literaturze aktuarialnej według, której rezerwę oblicza się jako wartość aktuarialną zakumulowanych przyszłych przepływów pieniężnych uwzględniając ryzyko śmierci i zmianę wartości pieniądza w czasie czyli tzw. ryzyko aktuarialne. Firmy ubezpieczeniowe oferujące złożone produkty ubezpieczeniowe jakimi są ubezpieczenia na życie z opcjami dodatkowymi m.in z opcją ADBs (Accelerated Death Benefits), ALIs (Acceleration Life Insurance) i ADIs (Accidental

Death Insurance), zgodnie z SOLVENCY II powinny uwzględniać w kalkulacjach również dodatkowy aspekt ryzyka wynikający z rozszerzonego ryzyka aktuarialnego objętego ochroną ubezpieczeniową W artykule łącząc podejście finansowe i ubezpieczeniowe przeprowadzono kalkulację rezerw dla ubezpieczenia na życie z opcją dodatkową typu ADBs, ALIs i ADIs którą wyznaczono jako odpowiednią warunkową wartość oczekiwaną z uwzględnieniem rozszerzonego ryzyka aktuarialnego oraz zbadano wpływ dodatkowych opcji na ich wysokość.

Słowa kluczowe: wieloopcyjne ubezpieczenia na życie, ubezpieczeniowe opcje dodatkowe typu ABDs, ALI i ADI, wypłacalność, rezerwy matematyczne składek.

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